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ON ONE-DIMENSIONAL COUPLED DIRAC EQUATIONS(1)

BY

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ABSTRACT. The Cauchy Problem for Dirac equations coupled through scalar and Fermi interactions is considered in one space dimension. Global solutions of finite energy are shown to exist, provided that either the magnitude of the coupling constant or the $L_2(R^1)$-norm of the initial data is suitably restricted.

1. **Introduction.** We consider here global existence of solutions to Dirac equations coupled through a "scalar" interaction

$$U_t = \alpha U_x + M_1 \beta U - ig(V^* \beta V)\beta U,$$
$$V_t = \alpha V_x + M_2 \beta V - ig(U^* \beta U)\beta V, \tag{1.1}$$

and through a "Fermi" interaction

$$U_t = \alpha U_x + M_1 \beta U - g(|V|^2 + V^* \alpha V)\beta U,$$
$$V_t = \alpha V_x + M_2 \beta V - g(|U|^2 - U^* \alpha U)\beta V. \tag{1.2}$$

Here the spinors U, V are two-component column vectors with components $U^{(1)}$, $U^{(2)}$, $V^{(1)}$, $V^{(2)}$, respectively. U^* denotes the conjugate transpose of U, and the 2×2 matrices α, β satisfy $\alpha^* = \alpha$, $\alpha^2 = I$, $\beta^* = -\beta$, $\beta^2 = -I$, $\alpha\beta + \beta\alpha = 0$.

We will be interested in establishing the existence of global finite energy solutions of the above systems, that is, spinors U, V with

$$|U(t)|_{H^1}^2 = |U(t)|_2^2 + |U_x(t)|_2^2 < \infty$$

for each $t \geqslant 0$, with the same holding for V. The norms appearing on the right above are taken in $L_2(R^1)$.

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The three-dimensional versions of the above equations have been treated by Chadam [2]. There, global existence and a scattering theory were established, provided that either the coupling constant g or certain norms of the initial data were suitably small. The proof depended on sufficiently rapid temporal decay of free solutions (i.e. solutions of the linear Dirac equation) and of the Dirac propagator. In the case of one space dimension, these quantities do not seem to decay "fast enough" for the proof of [2] to apply. A second motivation for the present work is that the methods recently developed for the Maxwell-Dirac equations (cf. [1], [5]) do not apply. The reason is that for the Maxwell-Dirac equations the interaction is "quadratic", in contrast to the "cubic" interaction observed in (1.1), (1.2) above. Consequently, the estimates from [1] or [5], when applied to (1.1), (1.2), contain "too many derivatives" and cannot be used for global existence. Yet another interesting aspect of these equations is the indefiniteness of the "energy integral". Both of the above systems possess an invariant, called the "energy". It may be easily verified that, for (1.1), we have

$$\int_{R^1} [U^* \alpha U_x + V^* \alpha V_x + M_1 U^* \beta U + M_2 V^* \beta V - ig(U^* \beta U)(V^* \beta V)]\, dx = \text{const.}$$

Due to the indefinite nature of the last term in the integrand, it is not at all clear that any estimates can be derived from this. Even if the last term were to enjoy a "positivity" property, any such resulting estimate, as is easily seen, would be no stronger than that obtainable directly from the Sobolev inequality in one dimension. It is interesting to note that this also occurs for certain versions of the three-dimensional Maxwell-Dirac equations [4]. On the other hand, the a priori estimate in one dimension for the Maxwell-Dirac and Klein-Gordon-Dirac equations, derived from the energy integral, has yielded global existence [5].

These peculiarities have prompted the present work in which we shall prove (§2) the global existence of finite energy solutions to (1.1) and (1.2), provided that either the magnitude of the coupling constant g or the $L_2(R^1)$-norm of the Cauchy data is sufficiently small. The key estimate is a uniform bound, which is obtained from an additional integral representation of the solution resulting from integration of the systems along characteristics. Of importance here is the "cone estimate" (Lemma 2.1). It is remarkable that the seemingly weak conservation of charge relationship (equation (2.4), §2), from which the cone estimate is derived, is strong enough to yield an existence proof.

All integrals to which no domain of integration is attached are taken over R^1, and we shall write $|u(t)|_p$ for the spatial $L_p(R^1)$-norm of $u(x, t)$, etc.

2. **Global existence.** We shall find below that the two systems (1.1), (1.2) are, with respect to existence, essentially the same. Therefore we shall first treat the Fermi system (1.2), returning later to (1.1). It will be convenient to work with the particular representations

$$\alpha_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \beta_0 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

of the Dirac matrices. This is justified since all representations of the Dirac matrices are unitarily equivalent. We diagonalize α_0 and define

$$\begin{aligned} u_1 &= U^{(1)} + U^{(2)}, & u_2 &= U^{(1)} - U^{(2)}, \\ v_1 &= V^{(1)} + V^{(2)}, & v_2 &= V^{(1)} - V^{(2)}, \end{aligned} \tag{2.2a}$$

and the corresponding vectors

$$u^* = (\bar{u}_1, \bar{u}_2), \quad v^* = (\bar{v}_1, \bar{v}_2). \tag{2.2b}$$

Then (2.1) becomes the system

$$\begin{aligned} u_t &= Au_x + M_1 Bu - g|v_2|^2 Bu, \\ v_t &= Av_x + M_2 Bv - g|u_1|^2 Bv, \end{aligned} \tag{2.3}$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is clear that A, B satisfy the same symmetry and anticommutation properties as those previously demanded of α, β, respectively.

The treatment of local existence of finite energy solutions (i.e. solutions with $|u(t)|_{H^1}$, $|v(t)|_{H^1}$ finite at each t) is well known (cf. [8]). One inverts the linear part of (2.3) and obtains a nonlinear integral equation for the solutions u, v. The spatial integral so obtained is a convolution of the Dirac propagator with the nonlinearity, which is here essentially "cubic". In view of the one-dimensional Sobolev inequality, such nonlinearities are Lipschitz on $H^1(R^1)$, so that the usual successive approximations converge locally in time over $H^1(R^1)$ to a unique solution. Global existence will follow once we show that the $H^1(R^1)$-norm of the solution remains finite for all $t \geqslant 0$ (cf. [8]). The rest of this section is devoted to obtaining such estimates. For simplicity, we will assume the Cauchy data are smooth and vanish sufficiently rapidly at infinity.

We begin with the *conservation of charge* identity

(2.4) $$(u^* u)_t = (u^* A u)_x, \quad (v^* v)_t = (v^* A v)_x.$$

These relationships are easily obtained as follows: one left-multiples the first of equations (2.3), say, by u^*, and takes the real part of the resulting expression. The first of equations (2.4) follows, when we take into account the symmetry property $B^* = -B$. Obviously the second equation of (2.4) is obtained in the same fashion. It follows from (2.4) that any solution satisfies

(2.5) $$|u(t)|_2 = |u(0)|_2, \quad |v(t)|_2 = |v(0)|_2.$$

We further exploit the conservation law (2.4) by proving the following, which we call the "cone estimate":

LEMMA 2.1. *For arbitrary x, t we have*

$$2\int_0^t [|u_1(x+t-s,s)|^2 + |u_2(x-t+s,s)|^2]\,ds \leqslant |u(0)|_2^2,$$

$$2\int_0^t [|v_1(x+t-s,s)|^2 + |v_2(x-t+s,s)|^2]\,ds \leqslant |v(0)|_2^2.$$

PROOF. Since both estimates follow from (2.4), it will suffice to establish the first. Using the explicit form of A we get

$$u^* A u = -|u_1|^2 + |u_2|^2$$

so that (2.4) takes the form

(2.6) $$0 = (|u_1|^2 + |u_2|^2)_t - (-|u_1|^2 + |u_2|^2)_x.$$

Let (x,t) with $t > 0$ be an arbitrary point, and let $C = C(x,t)$ be the backward characteristic cone through (x,t): $C(x,t) = \{(y,s)\colon |y-x| \leqslant t - s, 0 \leqslant s \leqslant t\}$. Then we integrate (2.6) over C and use Green's Theorem. This gives

$$0 = \int_{\mathrm{bdy}\,C} \left(|u_1|^2 + |u_2|^2\right) dy + \left(|u_2|^2 - |u_1|^2\right) ds$$

with the boundary of C, bdy C, given by bdy $C = B_0 \cup C_1 \cup C_2$ where B_0 is the interval $[x-t, x+t]$ on $s = 0$, and $C_1 = \{(y,s)\colon y = x+t-s, 0 \leqslant s \leqslant t\}$, $C_2 = \{(y,s)\colon y = x-t+s, 0 \leqslant s \leqslant t\}$. Thus an elementary calculation shows that

$$2\int_0^t [|u_1(x+t-s,s)|^2 + |u_2(x-t+s,s)|^2]\,ds = \int_{x-t}^{x+t} |u(y,0)|^2\,dy$$

from which the lemma follows.

The next lemma provides the crucial estimate, a uniform bound. As above, let $C = C(x,t)$ be the backward characteristic cone through an arbitrary point (x,t) with $t > 0$. We write out the system (2.3) in components to get

$$\begin{aligned} u_{1_t} &= -u_{1_x} - iM_1 u_2 + ig|v_2|^2 u_2, \qquad & v_{1_t} &= -v_{1_x} - iM_2 v_2 + ig|u_1|^2 v_2, \\ u_{2_t} &= u_{2_x} - iM_1 u_1 + ig|v_2|^2 u_1, \qquad & v_{2_t} &= v_{2_x} - iM_2 v_1 + ig|u_1|^2 v_1. \end{aligned} \tag{2.7}$$

Since A is diagonal, the system (2.3) (and thus (2.7)) can be integrated on characteristics (whose equations are $dx/dt = \pm 1$) to yield the following integral representations:

(2.8)

$$u_1(x,t) = u_{10}(x-t) + i\int_0^t [-M_1 + g|v_2(x-t+s,s)|^2] u_2(x-t+s,s)\,ds,$$

$$u_2(x,t) = u_{20}(x+t) + i\int_0^t [-M_1 + g|v_2(x+t-s,s)|^2] u_1(x+t-s,s)\,ds,$$

$$v_1(x,t) = v_{10}(x-t) + i\int_0^t [-M_2 + g|u_1(x-t+s,s)|^2] v_2(x-t+s,s)\,ds,$$

$$v_2(x,t) = v_{20}(x+t) + i\int_0^t [-M_2 + g|u_1(x+t-s,s)^2] v_1(x+t-s,s)\,ds.$$

Here u_{10}, u_{20}, v_{10}, v_{20} are the initial values of u_1, u_2, v_1, and v_2, respectively. Now we have

LEMMA 2.2. *Let the condition $g|u(0)|_2|v(0)|_2 < 2$ be satisfied. Then there exist continuous functions F, G such that*

$$\sup_x |u(x,t)| \leqslant F(t), \quad \sup_x |v(x,t)| \leqslant G(t) \quad \textit{for all } t \geqslant 0.$$

PROOF. From the first of equations (2.8) we have

$$\begin{aligned} |u_1(x,t)| &\leqslant |u_{10}(x-t)| + M_1 \int_0^t |u_2(x-t+s,s)|\,ds \\ &\quad + g\int_0^t |v_2(x-t+s,s)|^2 |u_2(x-t+s,s)|\,ds \\ &\leqslant \sup_x |u_{10}(x)| + M_1 t^{1/2} \left(\int_0^t |u_2(x-t+s,s)|^2\,ds\right)^{1/2} \\ &\quad + g\left(\sup_{\substack{s \\ 0 \leqslant s \leqslant t}} |v_2(x-t+s,s)|\right)\left(\int_0^t |v_2(x-t+s,s)|^2\,ds\right)^{1/2} \\ &\quad \times \left(\int_0^t |u_2(x-t+s,s)|^2\,ds\right)^{1/2} \end{aligned}$$

where we have used the Schwarz inequality. Now the square integrals of u_2, v_2 here can be estimated above using Lemma 2.1. We thus obtain

$$|u_1(x,t)| \leqslant \sup_x |u_{10}(x)| + M_1 |u(0)|_2 (t/2)^{1/2} + (g/2)(|u(0)|_2 |v(0)|_2) \sup_{0 \leqslant s \leqslant t} |v_2(x-t+s,s)|.$$

But the supremum of v_2 above is certainly dominated by $\sup_{(y,s)\in C}|v_2(y,s)|$, which we shall denote simply by $\sup_C |v_2|$. Thus we have

$$|u_1(x,t)| \leqslant \sup_x |u_{10}(x)| + M_1 |u(0)|_2 (t/2)^{1/2} + (g/2)(|u(0)|_2 |v(0)|_2) \sup_C |v_2|. \tag{2.9}$$

Now a domain of dependence argument (following from the representation (2.8)) shows that (2.9) is valid for every point $(x_0, t_0) \in C$. Hence we find from (2.9) that

$$\sup_C |u_1| \leqslant \sup_x |u_{10}(x)| + M_1 |u(0)|_2 (t/2)^{1/2} + (g/2)(|u(0)|_2 |v(0)|_2) \sup_C |v_2|. \tag{2.10}$$

We now estimate the last of equations (2.8) in the same manner. This yields

$$\begin{aligned} |v_2(x,t)| &\leqslant |v_{20}(x+t)| + M_2 t^{1/2} \left(\int_0^t |v_1(x+t-s,s)|^2 ds \right)^{1/2} \\ &\quad + g \left(\sup_{0 \leqslant s \leqslant t} |u_1(x+t-s,s)| \right) \left(\int_0^t |u_1(x+t-s,s)|^2 ds \right)^{1/2} \\ &\quad \times \left(\int_0^t |v_1(x+t-s,s)|^2 ds \right)^{1/2} \\ &\leqslant \sup_x |v_{20}(x)| + M_2 |v(0)|_2 (t/2)^{1/2} + (g/2) |u(0)|_2 |v(0)|_2 \sup_C |u_1|. \end{aligned}$$

Therefore we have

$$\sup_C |v_2| \leqslant \sup_x |v_{20}(x)| + M_2 |v(0)|_2 (t/2)^{1/2} + (g/2) |u(0)|_2 |v(0)|_2 \sup_C |u_1|. \tag{2.11}$$

We now combine (2.10) and (2.11) to get

$$\sup_C |u_1| \leqslant \sup_x |u_{10}(x)| + M_1 |u(0)|_2 (t/2)^{1/2} \\ +(g/2)|u(0)|_2 |v(0)|_2 \sup_x |v_{20}(x)| \; +(g/2)|u(0)|_2|v(0)|_2 \\ \times \left[M_2 |v(0)|_2 (t/2)^{1/2} + (g/2)|u(0)|_2 |v(0)|_2 \sup_C |u_1| \right].$$

Obviously a bound on $\sup_C |u_1|$ results, provided $(g^2/4)|u(0)|_2^2 |v(0)|_2^2 < 1$, that is, provided the hypothesis of the lemma is satisfied. Given this bound (call it $F_1(t)$ on $\sup_C|u_1|$) we find a bound $G_2(t)$, say, on $\sup_C|v_2|$ from (2.11). Then from the second of equations (2.8) we have

$$|u_2(x,t)| \leqslant \sup_x |u_{20}(x)| + M_1 |u(0)|_2 \left(\frac{t}{2}\right)^{1/2} + g\int_0^t G_2^2(s) F_1(s)\,ds,$$

and the third of equations (2.8) yields

$$|v_1(x,t)| \leqslant \sup_x |v_{10}(x)| + M_2 |v(0)|_2 \left(\frac{t}{2}\right)^{1/2} + g\int_0^t F_1^2(s) G_2(s)\,ds.$$

Both of these expressions are finite for all x, t, and this proves the lemma.

After these preliminaries, we can now prove

THEOREM 2.1. *The Cauchy Problem for the coupled Dirac system with "Fermi" interaction* (2.3) *has a unique, global, finite energy solution, provided* $g|u(0)|_2 \cdot |v(0)|_2 < 2$.

PROOF. We need only show that the norms $|u_x(t)|_2$, $|v_x(t)|_2$ are finite at each $t \geqslant 0$. We now differentiate the first of equations (2.3) with respect to x, and left-multiply the result by u_x^* to find

$$u_x^* u_{tx} = u_x^* A u_{xx} + M_1 u_x^* B u_x - g(|v_2|^2)_x u_x^* B u - g|v_2|^2 u_x^* B u_x.$$

We conjugate this, add the result to the above equation, and integrate over x to get

$$\frac{d}{dt}|u_x(t)|_2^2 = 2g\,\mathrm{Re}\int (|v_2|^2)_x u^* B u_x\,dx$$

since $B^* = -B$. An entirely similar calculation produces

$$\frac{d}{dt}|v_x(t)|_2^2 = 2g\,\mathrm{Re}\int (|u_1|^2)_x v^* B v_x\,dx.$$

It follows that

$$|u_x(t)|_2^2 + |v_x(t)|_2^2 \leqslant |u_x(0)|_2^2 + |v_x(0)|_2^2$$
$$+4g\int_0^t \int [|v_2|\,|(v_2)_x|\,|u^* B u_x| + |u_1|\,|(u_1)_x|\,|v^* B v_x|]\,dx\,ds$$
$$\leqslant \text{const.}\left[1 + \int_0^t \sup_x |u(x,s)| \sup_x |v(x,s)|\,|u_x(s)|_2 |v_x(s)|_2\,ds\right]$$

where we have used the Schwarz inequality. Applying the estimates of Lemma 2.2 we get

$$|u_x(t)|_2^2 + |v_x(t)|_2^2 \leqslant \text{const.}\left(1 + \int_0^t F(s)G(s)\,|u_x(s)|_2 |v_x(s)|_2\,ds\right)$$
$$\leqslant \text{const.}\left(1 + \int_0^t F_1(s)(|u_x(s)|_2^2 + |v_x(s)|_2^2)\,ds\right)$$

where $F_1(s)$ is a continuous function. Now Gronwall's inequality is applicable, and yields the finiteness of the norms $|u(t)|_{H^1}$, $|v(t)|_{H^1}$ for each $t \geqslant 0$, proving the theorem.

We now turn to the scalar-coupled Dirac system (1.1). With the same notation as (2.2a), (2.2b), we rewrite that system as

$$\text{(2.12)} \qquad \begin{aligned} u_t &= Au_x + M_1 Bu - g(iV^* \beta_0 V)Bu, \\ v_t &= Av_x + M_2 Bv - g(iU^* \beta_0 U)Bv. \end{aligned}$$

Now

$$iU^* \beta_0 U = |U^{(1)}|^2 - |U^{(2)}|^2 = |\tfrac{1}{2}(u_1 + u_2)|^2 - |\tfrac{1}{2}(u_1 - u_2)|^2 = \operatorname{Re} u_1 \bar{u}_2$$

and, similarly, $iV^* \beta_0 V = \operatorname{Re} v_1 \bar{v}_2$. The integral equations which correspond to (2.8) for this system become

(2.13)

$$u_1(x,t) = u_{10}(x - t) + i\int_0^t [-M_1 + g \operatorname{Re} v_1 \bar{v}_2(x - t + s, s)]u_2(x - t + s, s)\,ds,$$

$$u_2(x,t) = u_{20}(x + t) + i\int_0^t [-M_1 + g \operatorname{Re} v_1 \bar{v}_2(x + t - s, s)]u_1(x + t - s, s)\,ds,$$

$$v_1(x,t) = v_{10}(x - t) + i\int_0^t [-M_2 + g \operatorname{Re} u_1 \bar{u}_2(x - t + s, s)]v_2(x - t + s, s)\,ds,$$

$$v_2(x,t) = v_{20}(x + t) + i\int_0^t [-M_2 + g \operatorname{Re} u_1 \bar{u}_2(x + t - s, s)]v_1(x + t - s, s)\,ds.$$

We note that the result of Lemma 2.1 is still valid, since its proof depended only on the conservation of charge (2.4). Now the nonlinearities here are again "cubic". Hence it is clear that the estimate on $|u_x(t)|_2$, $|v_x(t)|_2$ will go through as before, provided that a uniform estimate of the form presented in Lemma 2.2 holds. To derive such an estimate, we treat the first and third of equations (2.13) above as before to find a bound on $\sup_C |u_1|$, and hence a bound on $\sup_C |v_1|$, provided the condition $g|u(0)|_2|v(0)|_2 < 2$ is met. It is clear that the second and the fourth of equations (2.13) may be similarly treated to give uniform bounds on u_2 and v_2. We have thus proved the following:

THEOREM 2.2. *The Cauchy Problem for the scalar-coupled Dirac equations* (2.12) *has a unique, global, finite energy solution, provided* $g|u(0)|_2|v(0)|_2 < 2$.

REFERENCES

1. J. Chadam, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension*, J. Functional Analysis **13** (1973), 173–184.
2. ———, *Asymptotic behavior of equations arising in quantum field theory*, Applicable Anal. **3** (1973), 377–402.
3. J. Chadam and R. Glassey, *On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions*, Arch. Rational Mech. Anal. **54** (1974), 223–237.
4. ———, *On the Maxwell-Dirac equations with zero magnetic field and their solution in two space dimensions*, J. Math. Anal. Appl. **53** (1976), 495–507.
5. ———, *New estimates for the global solution of the one-dimensional Maxwell-Dirac and Klein-Gordon-Dirac equations* (to appear).
6. R. Glassey, *On the asymptotic behavior of nonlinear wave equations*, Trans. Amer. Math. Soc. **182** (1973), 187–200. MR **48** #9119.
7. J. Keller, *On solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **10** (1957), 523–530. MR **20** #3371.
8. I. Segal, *Non-linear semi-groups*, Ann. of Math. (2) **78** (1963), 339–364. MR **27** #2879.

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